Computational Complexity III: Limits of Computation

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Context

Computability vs Complexity

Computability

What can be computed and what can not be computed?

Complexity

What can be computed fast and what can not be computed?

Computability vs Complexity

Computability

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Complexity

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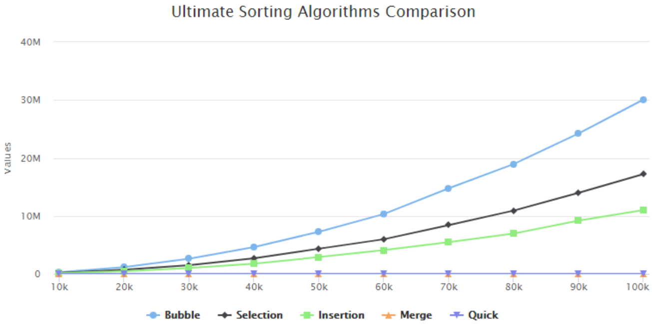

Figure: Comparison of sorting algorithms

Time Complexity of DTM

Definition

Let t: $\mathbb{N} \longrightarrow \mathbb{N}$ increasing function. The **time complexity** of DTIME[t(n)] is the collection of all languages that are decidable by an $O(t(n))$ time DTM.

DTIME[t(n)] ≡ { P: P is solved in O(t(n)) time }

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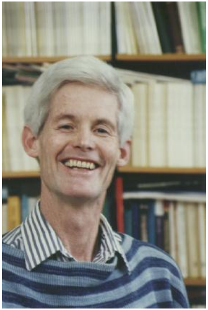
Definition

Complexity class $\mathcal{P}$ is the set of decision problems that can be solved by a DTM in a polynomial time of steps.

$\mathcal{P} \equiv \bigcup_{k \geq 0} DTIME[n^k]$

Cook - Karp Thesis

The **Cook - Karp Thesis** states that decision problems that are "tractably computable" can be computed by a DTM in polynomial time, i.e., are in $\mathcal{P}$.



Time Complexity of NTM

Definition

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Definition

Complexity class $\mathcal{NP}$ is the set of decision problems that can be solved by a NTM in a polynomial time of steps or is the set of decision problems for which there exists a **poly time certifier**.

$\mathcal{NP} \equiv \bigcup_{k\geq 0} NTIME[n^k]$

$\mathcal{P}$ vs $\mathcal{NP}$

How much easier is to **find** a solution than to **confirm** it?

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$\mathcal{P} \subseteq \mathcal{NP}$

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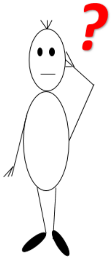

Open Problem: $\mathcal{P} \supseteq^{???} \mathcal{NP}$

TSP $\in \mathcal{NP}$

Travelling Salesman Problem (TSP): Given a set of distances on n cities and a bound D, is there a tour of length at most D?

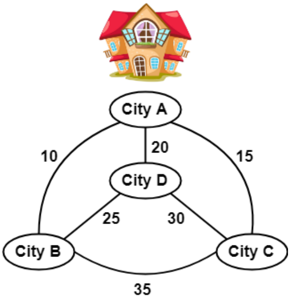

Figure: TSP

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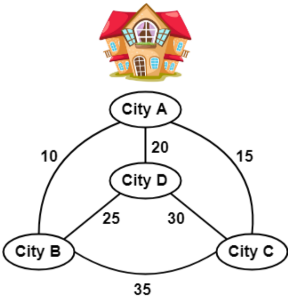


Figure: TSP

Certificate: A tour of given graph.

Certifier:

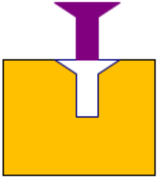
1. Check that each city appears once.
2. Check that the length of tour is at most D.

Context

Polynomial time reduction

Figure: The casting process

Polynomial time reduction



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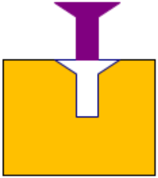

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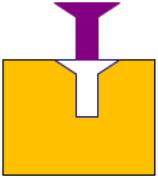

Figure: The casting process

reduction →

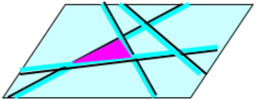

Figure: Half plane intersection

Polynomial time reduction

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Hamiltonian Cycle Problem $\xrightarrow{\textit{reduction}}$ Travelling Salesman Problem

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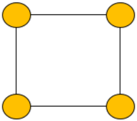

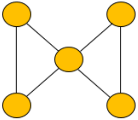

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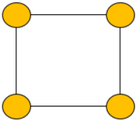

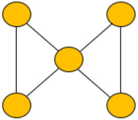

Travelling Salesman Problem

Let $G' = (V', E')$ a weighted graph with non negative weights and $k' \in \mathbb{Z}$.
Find whether G' contains a cycle that passes through all vertices of the graph exactly once and has length $\leq k'$.

Goal of the study of $\mathcal{NP}$ - completeness

If some $\mathcal{NP}$ - complete problem P is in $\mathcal{P}$, then $\mathcal{P} = \mathcal{NP}$.

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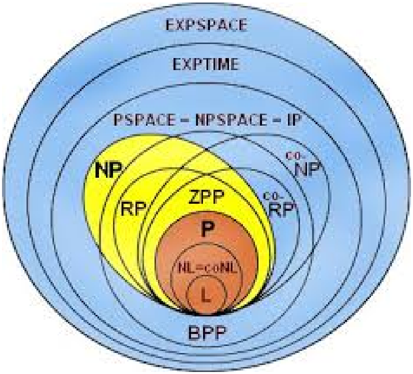

Figure: Scott Aaronson

Context

Definition

Let s: $\mathbb{N} \longrightarrow \mathbb{N}$ increasing function. The **space complexity** of DSPACE[t(n)] is the collection of all languages that are decidable by an $O(s(n))$ space DTM.

NSPACE[s(n)] ≡ { P: P is solved in O(s(n)) space}

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Definition

Complexity class *PSPACE* is the set of decision problems that can be solved by a (multitape) DTM in a polynomial number of SPACEs on the tape.
$PSPACE \equiv \bigcup_{k \geq 0} DSPACE[n^k]$

Theorem

$\mathcal{P} \subseteq PSPACE$

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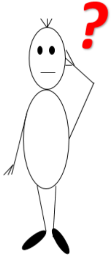
Open Problem:
$\mathcal{P} \supseteq^{???} \mathcal{NP} \supseteq^{???} PSPACE$



PSPACE-complete

Definition

$X \in PSPACE$-complete if:

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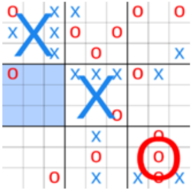
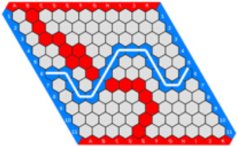
- $\forall\ Y \in PSPACE,\ Y \leq_P X$

GAMES



ultimate tic tac toe



hex

go

Figure: PSPACE-complete problems

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Thank you!